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Wall Crossing of BPS States on the Conifold from Seiberg Duality and Pyramid Partitions

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Abstract

In this paper we study the relation between pyramid partitions with a general empty room configuration (ERC) and the BPS states of D-branes on the resolved conifold. We find that the generating function for pyramid partitions with a length n ERC is exactly the same as the D6/D2/D0 BPS partition function on the resolved conifold in particular Kähler chambers. We define a new type of pyramid partition with a finite ERC that counts the BPS degeneracies in certain other chambers.

The D6/D2/D0 partition functions in different chambers were obtained by applying the wall crossing formula. On the other hand, the pyramid partitions describe T^3 fixed points of the moduli space of a quiver quantum mechanics. This quiver arises after we apply Seiberg dualities to the D6/D2/D0 system on the conifold and choose a particular set of FI parameters. The arrow structure of the dual quiver is confirmed by computation of the Ext group between the sheaves. We show that the superpotential and the stability condition of the dual quiver with this choice of the FI parameters give rise to the rules specifying pyramid partitions with length n ERC.

1 Introduction

One of the most fruitful areas of overlap between string theory and mathematics has been in the applications of topological field theories and string theories to questions involving integration over various moduli spaces of interesting geometrical objects. One example is that the topological sector of the worldvolume theory of a D6 brane wrapping a Calabi-Yau 3-fold has been identified with Donaldson-Thomas theory. The bound states of D2

and D0 branes to a D6 brane can be regarded as instantons in the topologically twisted $\mathcal{N} = 2\, U(1)$ Yang-Mills theory in six dimensions. These instantons turn out to correspond to ideal sheaves on the 3-fold. Moreover, Donaldson-Thomas theory involves integration of a virtual fundamental class over the moduli space of such ideal sheaves.

There has recently been tremendous progress in understanding the Kähler moduli dependence of the index of BPS bound states of D-branes wrapping cycles in a Calabi-Yau 3-fold [1]. On the mathematical side, various generalizations of Donaldson-Thomas theory have been proposed [2], in which ideal sheaves are replaced by more general stable objects in the derived category. These invariants will thus also have a dependence on the background Kähler moduli, as encoded in a stability condition.

Much work has been done on Donaldson-Thomas theory in toric Calabi-Yau manifolds, which are of necessity noncompact. In particular, using toric localizations, the theory can be solved exactly [3, 4]. In this work we will focus on the particular example of the resolved conifold. More recently, there was some extremely interesting work [5, 6], in which the Donaldson-Thomas invariants on a noncommutative resolution of the conifold were determined. In that case, the torus fixed points of the moduli space of noncommutative ideal sheaves were identified with pyramid partitions in a length 1 ERC.

In [7], these results were reproduced using the physical techniques resulting from the supergravity description of such bound states as multi-centered black holes in IIA string theory compactified on a Calabi-Yau manifold. The Donaldson-Thomas partition function of the commutative and noncommutative resolutions of the conifold were shown to arise as special cases in the moduli space of asymptotic Kähler parameters. Moreover, the D6/D2/D0 partition function was determined in all chambers of a certain real three parameter moduli space that captured the relevant universal behavior as a compact Calabi-Yau threefold degenerated to the noncompact resolved conifold[1].

In this work, we will demonstrate an intriguing relationship between the pyramid partitions in a length n ERC and the D6/D2/D0 partition function in various chambers. In other chambers, we will give evidence that the torus fixed points of the moduli space of BPS states are in one to one correspondence with a new type of pyramid partitions in a finite region.

It is possible to find a basis of D-brane charges which is both primitive (ie. it generates

[1]These partition functions were also derived first in [18] using a different approach.

the entire lattice) and rigid (ie. the basis branes have no moduli) for D6/D2/D0 branes in the resolved conifold. Note that this is never possible in compact Calabi-Yau. Thus the BPS bound states are completely described by the topological quiver quantum mechanics, whose fields are the open strings stretched between the basis branes. The quiver that describes D2/D0 in the conifold is in fact the famous $U(N) \times U(M)$ Klebanov-Witten quiver [8], viewed as a 0-dimensional theory in our context.

The new ingredient that we introduce is an extended quiver that also includes a $U(1)$ node associated to the D6 brane, in analogy with the case of D6/D0 bound states in $\mathbb{C}^3$ studied in [9]. We determine the spectrum of bifundamental strings by computing the appropriate *Ext* groups. One beautiful feature of this system is that the $SU(2) \times SU(2)$ symmetry of the conifold completely fixes the superpotential, SW, up to field redefinition, so we do not need to compute it directly. The moduli space of vacua of this quiver theory depends on the background Kähler moduli through the Fayet-Iliopoulos parameters.

This moduli space is obtained by imposing the F-term equations, $\partial W = 0$, on the Kähler quotient of the space of fields (here a finite dimensional space of matrices with ranks determined by the D-brane charges) by the $U(N) \times U(M) \times U(1)$ gauge group. The Kähler quotient is equivalent to the quotient by the complexified gauge group, together with an algebraic stability condition that depends on the FI parameters [10].

In each chamber of the Kähler moduli space of the resolved conifold found in [7], we identify a pair of primitive sheaves carrying D2/D0 charge that become mutually BPS at the boundary of the Kähler cone, in that chamber. In particular, along the boundary of the Kähler cone in the chamber $C_n = [\mathcal{W}_n^{-1}\mathcal{W}_{n-1}^{-1}]$, in the notation of [7] for $n > 0$, the central charges of $\mathcal{O}_C(-n-1)[1]$ and $\mathcal{O}_C(-n)$ become aligned. In the chamber $\tilde{C}_n = [\mathcal{W}_{n-1}^1\mathcal{W}_n^1]$, it is $\mathcal{O}_C(-n)$ and $\mathcal{O}_C(-n+1)[1]$ which become mutually BPS at the boundary of the Kähler cone.

Therefore in that chamber, we choose those sheaves, together with the pure D6 brane, $\mathcal{O}_X$, as our basis objects, and construct the quiver theory, as explained above. Note that all of the above pairs for different n are related to each other by application of Seiberg duality. We will denote the quivers resulting from the choice of basis branes above as Q_n and $\tilde{Q}_n$ respectively. The fact that the central charges become aligned in that chamber implies that the bifundamental strings between them are massless at tree level, hence the FI parameters, θ, for the two D2/D0 nodes must be equal for those values of the Kähler moduli.

It is easy to check that $\theta_{\mathcal{O}_X[1]} > 0$, $\theta_{\mathcal{O}_C(-n-1)[1]} < 0$, and $\theta_{\mathcal{O}_C(-n)} < 0$ is a single chamber, and includes the locus where $\theta_{\mathcal{O}_C(-n-1)[1]} = \theta_{\mathcal{O}_C(-n)}$. Therefore we can identify that chamber in the space of FI parameters for the quiver Q_n with the chamber C_n in the space of background Kähler moduli.

We show that with that choice of FI parameters, the King stability condition becomes equivalent to a simple cyclicity - the quiver representations must be generated by a vector in the $\mathbb{C}^1$ associated to the D6 brane $U(1)$ node. The relations obtained from the superpotential are used to see that the torus fixed points of the moduli space of stable representations in this chamber are exactly the pyramid partitions in a length n ERC defined in [5]! Similarly, for the chambers $\tilde{C}_n$, we demonstrate that the torus fixed points are in one to one correspondence with pyramid partitions in a certain finite empty room configuration that we introduce.

The generating functions for pyramid partition in a length n ERC was determined in [5, 6], and we check that it agrees with the D6/D2/D0 partition function in the resolved conifold in the appropriate chamber found in [7] [18] [19]. This requires correctly changing variables to take into account the D2/D0 charges of the basis sheaves used in the construction of the quivers. We explicitly check a few examples of the finite type pyramid partitions as well.

Thus we have been able to reproduce the D6/D2/D0 partition function on the resolved conifold, in a given chamber in the space of background Kähler moduli, by judiciously choosing a particular Seiberg dual version of the associated three node quiver in which the FI parameters corresponding to the Kähler moduli are of a special simple form.

The paper is organized as follows. In section 2 we give a review of wall crossing formulae and the D6/D2/D0 partition function on the resolved conifold based on [1, 7]. In section 3, we introduce the relation between the pyramid partition function and the D6/D2/D0 BPS partition function. In section 4 we derive the Seiberg dual quiver and then compute its superpotentials and arrow structures. Next the discussion is on the stability condition and the rules of pyramid partitions. Some future directions will be presented in the conclusion part.

2 Wall crossing formula and D6/D2/D0 BPS partition function on the resolved conifold

The index of BPS states with a given total charge is an integer, and thus is a piecewise constant function of the background values of the Kähler moduli. Moreover, the fact that it is a supersymmetric index implies that it can only jump when a state goes to infinity in the moduli space of BPS states, that is when the asymptotics of the potential change. The only known way this can happen for the case of BPS bound states of D-branes wrapping a Calabi-Yau manifold is that the physical size of a multi-centered Denef black hole solution diverges at some value of the Kähler parameters [11]. This occurs exactly at (real codimension 1) walls of marginal stability, when the central charges, Z_1 and Z_2, of the two constituents, Γ_1 and Γ_2, of the multi-centered supergravity solution become aligned.

At such a wall of marginal stability $t = t_{ms}$ corresponding to a decay $\Gamma \to \Gamma_1 + \Gamma_2$, the BPS index will have a discrete jump given by

$$\Delta\Omega(\Gamma, t) = (-1)^{\langle\Gamma_1,\Gamma_2\rangle - 1} \, |\langle\Gamma_1, \Gamma_2\rangle| \, \Omega(\Gamma_1, t_{ms})\Omega(\Gamma_2, t_{ms}), \tag{2.1}$$

where Γ_1 and Γ_2 are primitive. A semi-primitive wall crossing formula is also given in [1].

$$\Omega(\Gamma_1) + \sum_N \Delta\Omega(\Gamma_1 + N\Gamma_2)q^N = \Omega(\Gamma_1) \prod_{k>0} (1 - (-1)^{k\langle\Gamma_1,\Gamma_2\rangle} q^k)^{k|\langle\Gamma_1,\Gamma_2\rangle|\Omega(k\Gamma_2)}. \tag{2.2}$$

This formula gives a powerful way to construct the D6/D2/D0 BPS generating function on the resolved conifold from the Donaldson-Thomas generating function [7]. The absence of higher genus Gopakumar-Vafa invariants in the resolved conifold implies that only the pure D6 brane exists as a single centered solution. Thus in the core region of the Kähler moduli space, the D6/D2/D0 partition function is just $Z = 1$.

The position of the relevant walls of marginal stability was determined in [7], and using the wall crossing formula for a single D6 bound arbitrary numbers of D2/D0 fragments, the partition function was then computed throughout the moduli space. The wall of marginal stability for $\Gamma = 1 - m'\beta + n'dV$ with $\Gamma_h = -m_h\beta + n_h dV$ in a compact Calabi-Yau manifold, X, was shown have a well defined limit as the geometry approached that of the noncompact resolved conifold. Moreover, the walls are independent of m' and n', and

separate chambers in a real three dimensional space parameterized by the Kähler size, z, of the local $\mathbb{P}^1$, and a real variable $\varphi = \frac{1}{3}\arg(Vol_X)$ that characterizes the strength of the B-field along the noncompact directions in units of the Kähler form.

The wall of marginal stability for the fragment Γ_h was denoted by $\mathcal{W}_{n_h}^{m_h}$. The final result for the index of D6/D2/D0 bound states found in [7] was that in the chamber between $\mathcal{W}_n^1$ and $\mathcal{W}_{n+1}^1$, the generating function is

$$Z(u, v; [\mathcal{W}_n^1 \mathcal{W}_{n+1}^1]) = \prod_{j=1}^{n} \left(1 - (-u)^j v\right)^j . \tag{2.3}$$

Similarly, in the chambers where negative D2 charges appear,

$$Z(u, v; [\mathcal{W}_{n+1}^{-1} \mathcal{W}_n^{-1}]) = \prod_{j>0} \left(1 - (-u)^j\right)^{-2j} \left(1 - (-u)^j v\right)^j \prod_{k>n} \left(1 - (-u)^k v^{-1}\right)^k . \tag{2.4}$$

In the extreme case, $n = 0$, of the latter they found agreement with the results of Szendrői, who calculated the same partition function at the conifold point using equivariant techniques to find the Euler character of a moduli space of noncommutative sheaves. The $n \to \infty$ limit of (2.4), one obtains the usual "large radius" Donaldson-Thomas theory that was determined in [3], again using equivariant localization.

3 Pyramid partition and BPS partition function on resolved conifold

In this section we will first discuss the relation between the pyramid partition generating function with length n empty room configuration (ERC) and the D6/D2/D0 BPS state partition function. Afterwards we will discuss how the pyramid partition arises when we look at the torus fixed points on the moduli space $\mathcal{M}_v$ of representations of the Calabi-Yau algebra A for the conifold quiver.

3.1 Pyramid partition generating function

The use of pyramid partitions in this context first arose in [5]. Consider the arrangement of stones of two different colors (white and grey) as in Figure 1. For a generic ERC with length n, there will be n white stones on the zeroth layer.[2] On layer $2i$, there are

[2] We count the layers as the zeroth, 1st, 2nd, and so on.

$(n+i)(1+i)$ white stones, while on layer $2i+1$ there are $(n+i+1)(1+i)$ grey stones. When we write generating functions, the number of white stones will be counted the power of q_0, and the grey stones by q_1.

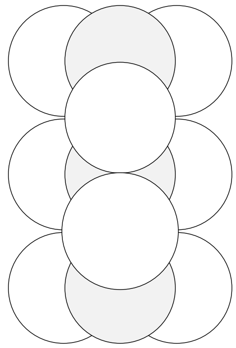

Figure 1: This figure illustrates the length 2 ERC of the pyramid partition, from the zeroth layer to the 2nd layer.

A finite subset Π of the ERC is a *pyramid partition*, if for every stone in Π, the stones directly above it are also in Π. Denote as w_0 and w_1 the number of white and grey stones in the partition. Also denote by $\mathcal{P}_n$ the set of all possible pyramid partitions for the ERC of length n.

The generating function is defined combinatorially as

$$Z_{pyramid}(n; q_0, q_1) = \sum_{\Pi \in \mathcal{P}_n} q_0^{w_0} (-q_1)^{w_1}. \tag{3.1}$$

This function can be computed by some dimer shuffling techniques; we refer the interested reader to [6] for details. Here we just quote the result for the generating function for general n ERC,

$$\begin{aligned} Z_{pyramid}(n; q_0, q_1) = M(1, -q_0 q_1)^2 & \prod_{k \geq 1} (1 + q_0^k (-q_1)^{k-1})^{k+n-1} \\ & \prod_{k \geq 1} (1 + q_0^k (-q_1)^{k+1})^{max(k-n+1,0)}, \end{aligned} \tag{3.2}$$

where $M(x, q)$ is the MacMahon function

$$M(x, q) = \prod_{n=1} (\frac{1}{1 - xq^n})^n. \tag{3.3}$$

Notice that the exponents of the two terms in the product formula start from n and 1 respectively. This function turns out to be exactly the D6/D2/D0 BPS partition function on the resolved conifold found in [7] [18] [19] in certain chambers after performing the following (n-dependent) parameter identifications:

$$u = -q_0 q_1 \ ; \ v = (-q_0)^{n-1} q_1^n. \tag{3.4}$$

Now we have

$$\begin{aligned} Z_{pyramid}(n; q_0, q_1) &= Z_{D6/D2/D0}(u, v; C_n) \\ &= M(1, -u) \prod_{k \geq 1} (1 - (-u)^k v)^k \prod_{k \geq n} (1 - (-u)^k v^{-1})^k. \end{aligned} \tag{3.5}$$

The upshot is that the pyramid partition for a general empty room configuration counts the number of D6/D2/D0 BPS bound states at a certain value of the background modulus! More precisely, this chamber sits between the conifold point and the large radius limit; $C_n = [\mathcal{W}_n^{-1} \mathcal{W}_{n-1}^{-1}]$ in the notation of [7].

We will explain that this is no coincidence, after we perform a Seiberg duality on the original D6/D2/D0 quiver theory. Moreover, the rules of specifying a pyramid partition encode the stability condition for these BPS states.

In the chamber $\tilde{C}_n = [\mathcal{W}_{n-1}^1 \mathcal{W}_n^1]$, the D6/D2/D0 BPS states partition function is given by

$$Z_{D6/D2/D0}(u, v, \tilde{C}_n) = \prod_{k=1}^{n-1} (1 - (-u)^{-k} v)^k. \tag{3.6}$$

We conjecture that the partition function in these chambers can be described by some *finite type* pyramid partitions with length $(n-1)$ ERC (see Figure 2 for length 3 example), after a change of variables.

For the finite type pyramid partition with length n, there are $n \times 1$ white stones on the zeroth layer, $(n-1) \times 1$ grey stones on the first layer, $(n-1) \times 2$ white stones on the second, $(n-2) \times 2$ grey stones on the third, and so on until we reach $1 \times n$. The way of counting is the same. We count the finite subsets Π of the ERC in which, for every stone in Π, the stones directly above it are also in Π.

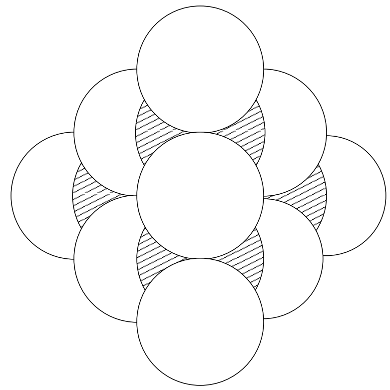

Figure 2: This is the ERC for the fintie type pyramid partition with length 3.

The change of variables is given by

$$u = -q_0 q_1, \quad v = -q_0^n q_1^{n-1}. \tag{3.7}$$

Let us consider length 2 ERC as an example to illustrate the correspondence between the BPS states partition function in chambers $\tilde{C}_n$ and the finite type pyramid partition.

$$\begin{aligned} Z_{n=2\ ERC} &= 1 + 2q_0 + q_0^2 + q_0^2 q_1 + 2q_0^3 q_1 + q_0^4 q_1 \\ &= (1 + q_0^2 q_1)(1 + q_0)^2 = (1 - (-u)^{-1} v)(1 - (-u)^{-2} v)^2 \\ &= Z(u^{-1}, v, \tilde{C}_3). \end{aligned} \tag{3.8}$$

3.2 Conifold quiver and pyramid partitions

This section is a review of [5]. Consider the conifold quiver $Q = \{V, E\}$, with two vertices $V = \{0, 1\}$, and four oriented edges $E = \{A_1, A_2 : 0 \to 1, B_1, B_2 : 1 \to 0\}$. The F-term relations come from the quartic superpotential $W = A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1$.[8]

The quiver algebra A contains the idempotent ring $\mathbb{C}[f_0, f_1]$ and can be given by generators and relations as

$$A = \mathbb{C}[f_0, f_1]\langle A_1, A_2, B_1, B_2\rangle / \langle B_1 A_i B_2 - B_2 A_i B_1, A_1 B_i A_2 - A_2 B_i A_1, i = 1, 2\rangle. \tag{3.9}$$

A is a smooth Calabi-Yau algebra of dimension three [12] and a crepant non-commutative resolution of the singularity Spec $(\mathbb{C}[x_1, x_2, x_3, x_4]/(x_1 x_2 - x_3 x_4))$.

Consider the rank two torus action T_W on the moduli space $\mathcal{M}_V$ of framed cyclic

A-modules [3]. It has been shown by Szendrői [5] that the T_W-fixed points on the moduli space $\mathcal{M}_V$ are all isolated and have a one-to-one correspondence with pyramid partitions $\Pi \in \mathcal{P}_1$ of weight (w_0, w_1). This weight vector is the same as the rank vector of the corresponding quiver.

Moreover, given a pyramid partition $\Pi \in \mathcal{P}_1$, we can obtain the precise framed cyclic module $\mathbb{M}_\pi$ defined by it from looking at the pyramid partition. First, we draw A_1 and A_2 fields in the perpendicular direction out of the center of the white stones and draw B_1 and B_2 fields in the horizontal direction out of grey stones. The superpotential F-term relations require that we get the same result if we follow the arrows of opposite directions of the A_i or B_i fields down to the three lower layers.

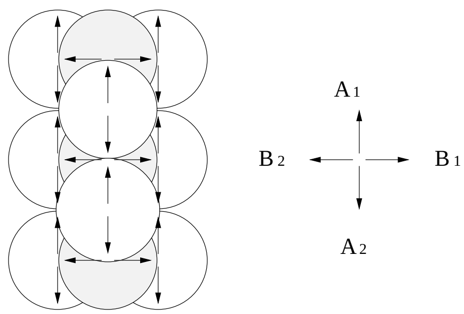

Figure 3: This figure illustrates how to define a framed cyclic module from a pyramid partition.

The cyclicity property of the module turns into the rule that for every stone in Π, the stones directly above it are also in Π. The module is generated by the stones on the zeroth layer. We show in section 5 that the cyclicity condition is equivalent to the King stability condition in a particular chamber of the quiver (Fig. 4) obtained by introducing a new node for the D6 brane to the Klebanov-Witten quiver discussed in [5].

In [5], Szendrői generalized the notion of pyramid partitions to length n ERC, and his conjecture for the resulting generating function was proven in [6]. We shall show that these partitions arise as torus fixed points of the moduli space of King stable representations of the quiver Q_n. For now, note that in the case of general length n ERC, the n stones on the zeroth layer will still play the roles of the framing vectors of the quiver and generate the whole module.

[3]The torus action fixing the superpotential is a rank three torus T_{FW}, described by $(A_1, A_2, B_1, B_2) \to (\mu^a A_1, \mu^b A_2, \mu^c B_1, \mu^{-a-b-c} B_2)$. And T_W is the quotient of T_{FW} by the C^* action $(\mu, \mu, \mu^{-1}, \mu^{-1})$.

There are also new $(n-1)$ relations for $n>1$, if we follow the arrows from the layer zero to layer one, which read:

$$A_1 q_2 = A_2 q_1,\ A_1 q_3 = A_2 q_2,\ \cdots, A_1 q_n = A_2 q_{n-1}, \tag{3.10}$$

where $q_1 \cdots q_n$ are the framing nodes on the layer zero. Later we will see that these relations arise from certain cubic terms in the superpotential which are not present for $n=1$.

4 Deriving the Quivers via Seiberg Duality

Recall that a standard choice of the sheaves representing the conifold quiver is as follows [13].

$$\mathcal{O}_X[1],\ \mathcal{O}_C,\ \mathcal{O}_C(-1)[1]. \tag{4.1}$$

The arrow structure of this quiver is determined by the Ext group.[4]

$$\mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}(-1)[1]) \cong \mathrm{Ext}^1(\mathcal{O}(-1)[1], \mathcal{O}_C) \cong \mathbb{C}^2. \tag{4.2}$$

So if we take the rank vector to be $(1, M+N, M)$, the system will have charges $(D6, D2, D0) = (1, M, N)$. Now we are going to show that this quiver at certain FI parameters leads to the pyramid partition after performing Seiberg dualities.

First of all we know that in the pyramid partition there are n marked framing nodes, which are the most top nodes $q_1 \cdots q_n$ and $n-1$ relations:

$$A_1 q_2 = A_2 q_1,\ A_1 q_3 = A_2 q_2,\ \cdots, A_1 q_n = A_2 q_{n-1}. \tag{4.3}$$

This implies that we want to find a quiver representation with the following arrow structure. (See Figure 4.)

For this purpose we choose the basis to be

$$\mathcal{O}_X,\ \mathcal{O}_C(-n-1),\ \mathcal{O}_C(-n)[-1]. \tag{4.4}$$

[4]We will use Ext and $\underline{Ext}$ to denote the global and local Ext respectively.

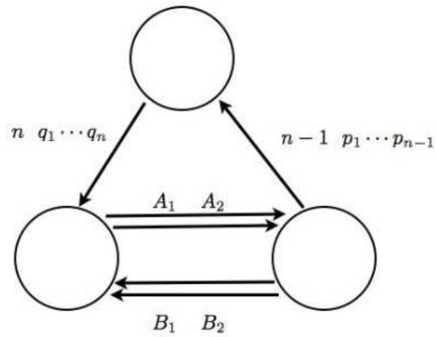

Figure 4: The n q fields gives n marked points in the quiver, which generates the whole module, and the $(n-1)$ p fields will give $(n-1)$ relation via superpotential. The sheaves representing the nodes are to be determined.

Or equivalently, by an overall shift,[5]

$$\mathcal{O}_X[1],\ \mathcal{O}_C(-n-1)[1],\ \mathcal{O}_C(-n): \qquad Q_n. \tag{4.5}$$

The quiver with this basis will be called Q_n.

Now let us try to find the sheaves corresponding to the finite type pyramid partition with length $n-1$. There are $n-1$ framing nodes on the top and n relations coming from the zeroth layer. So what we have to do is simply to reverse the directions of the p and q fields in Figure 4. The basis of the quiver is given by

$$\mathcal{O}_X[1],\ \mathcal{O}_C(-n-1),\ \mathcal{O}_C(-n)[1]: \qquad \tilde{Q}_{n-1}. \tag{4.6}$$

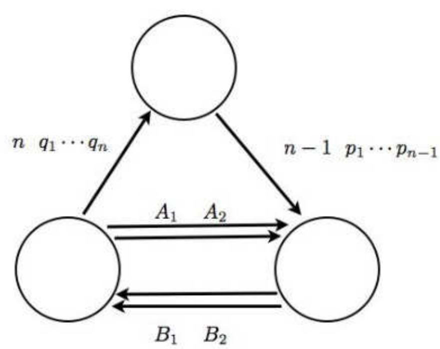

Figure 5: This shows the arrow structure of the quiver $\tilde{Q}_{n-1}$.

[5] In order to make contact with the convention in [13].

In the following section we will summarize the brane charges of the Seiberg dual quivers Q_n and then confirm the F-term relations imposed by superpotential and the arrow structures of the proposed quivers with Ext group computation. The Ext group computation for the quiver $\tilde{Q}_n$ simply follows from the computation for Q_n; therefore we only focus on Q_n from now on.

4.1 Brane charges of the quiver

The Chern character of the charges of the primitive objects in the derived category that we used as a basis in the quiver Q_n can be computed as

$$\big[\mathcal{O}_C(-n)\big] = -\beta + (1-n)dV, \qquad \big[\mathcal{O}_C(-n-1)[1]\big] = +\beta + ndV, \tag{4.7}$$

in the conventions of [1].

In the original D6/D2/D0 system, powers u and v count the D0 and D2 charges respectively. Suppose we have a bound state with 1 D6, M D2 and N D0 charges; this will be represented by a quiver with ranks determined by the following computation:

$$(-u)^N v^M = (q_0 q_1)^N ((-q_0)^{n-1} q_1^n)^M = (-1)^{n-1} q_0^{N+(n-1)M} q_1^{N+nM}. \tag{4.8}$$

Recall that q_0 are the number of white stones. Thus the ranks of the node $\mathcal{O}_C(-n-1)[1]$ and $\mathcal{O}_C(-n)$ are $N+(n-1)M$ and $N+nM$ respectively. This combination indeed gives the right total charges we are aiming at. We summarize the result in the following table. (See Table 1.)

Sheaves	Ranks	Charge	FI parameters
$\mathcal{O}_X[1]$	1	$D6$ or $\bar{D}6$	θ_1
$\mathcal{O}_C(-n-1)[1]$	$N+(n-1)M$	$\bar{D}2, nD0$	θ_2
$\mathcal{O}_C(-n)$	$N+nM$	$D2, (n-1)\bar{D}0$	θ_3

Table 1: Sheaves, ranks, and charges. Here we have taken into account the induced D0 charge of $\mathcal{O}_C$. The total charge of the system is M D2 and N D0.

4.2 $\mathcal{O}_X \to \mathcal{O}_C(-n-1)$

We now proceed to determine the number of bifundamental fields that appear in the quiver, by computing the *Ext* groups between the basis sheaves. First of all, since $\mathcal{O}_X$ is projective (thus free), we have

$$\underline{Ext}^i(\mathcal{O}_X, \mathcal{O}_C) = 0, \quad i > 0. \tag{4.9}$$

And we also have $\mathrm{Ext}^0(\mathcal{O}_X, \mathcal{O}_C(-n-1)) = 0$ when $n \geq 0$ since $\mathrm{Ext}^0(\mathcal{O}_X, \quad)$ is the global section functor. As for $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_C(-n-1))$, we need to use the following:

$$\dim \mathrm{H}^n(\mathbb{P}^n, \mathcal{O}(m)) = \binom{-m-1}{-n-m-1}. \tag{4.10}$$

Therefore,

$$\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_C(-n-1)) = \mathrm{H}^1(X, \mathcal{O}_C(-n-1)) \cong \mathbb{C}^n, \tag{4.11}$$

$$\mathrm{Ext}^2(\mathcal{O}_X, \mathcal{O}_C(-n-1)) = \mathrm{Ext}^3(\mathcal{O}_X, \mathcal{O}_C(-n-1)) = 0. \tag{4.12}$$

4.3 $\mathcal{O}_C(-n)[-1] \to \mathcal{O}_X$

According to Chapter 5.3 in [14], the local sheaves $\underline{Ext}^k(\mathcal{O}_C, \mathcal{O}_X)$ are all trivial except for $k = 2$.

$$\underline{Ext}^2(\mathcal{O}_C, \mathcal{O}_X) \cong \iota_* \mathcal{K}_C = \iota_* \mathcal{O}_C(-2), \tag{4.13}$$

where $\mathcal{K}_C$ is the canonical bundle over P^1.

Twisting the sheaf by $\mathcal{O}_X(n)$, we have

$$\underline{Ext}^2(\mathcal{O}_C(n), \mathcal{O}_X) \cong \iota_* \mathcal{K}_C(-n) = \iota_* \mathcal{O}_C(-2-n). \tag{4.14}$$

There is a local to global spectral sequence which we can apply to get the Ext group. However, if n is large, we can simply apply Property 6.9 in [15] to get the Ext group we want.

The property says, if $\mathcal{O}_X(1)$ is a very ample invertible sheaf and $\mathcal{E}$ and $\mathcal{F}$ are coherent sheaves on X, there exist an integer n_0, depending on $\mathcal{E}$, $\mathcal{F}$ and i, such that for $n > n_0$,

$$\mathrm{Ext}^i(\mathcal{E}, \mathcal{F}(n)) = \Gamma(X, \underline{Ext}^i(\mathcal{E}, \mathcal{F}(n))). \tag{4.15}$$

So for $n >> 0$, we have

$$\mathrm{Ext}^1(\mathcal{O}_C(-n), \mathcal{O}_X) = \Gamma(X, \underline{Ext}^1(\mathcal{O}_C, \mathcal{O}_X(n))) = 0, \tag{4.16}$$

$$\mathrm{Ext}^2(\mathcal{O}_C(-n), \mathcal{O}_X) = \Gamma(X, \underline{Ext}^2(\mathcal{O}_C, \mathcal{O}_X(n))) = \Gamma(X, \iota_* \mathcal{O}_C(n-2)) \cong \mathbb{C}^{n-1}, \tag{4.17}$$

$$\mathrm{Ext}^1(\mathcal{O}_C(-n)[-1], \mathcal{O}_X) \cong \mathbb{C}^{n-1}. \tag{4.18}$$

Now we sum up the computation in a quiver diagram, in which we actually apply an overall shift. (See Figure 6.)

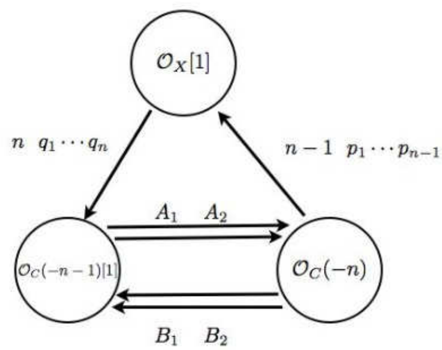

Figure 6: The quiver diagram for the pyramid partition with length n ERC.

4.4 Superpotential

In principle, the computation of the superpotential of this quiver quantum mechanics would require evaluating the B-model disk amplitude with boundary conditions determined given by the basis B-branes. Luckily that has already been done for the sheaves $\mathcal{O}_C(n)$ and $\mathcal{O}_C(m)[-1]$ by [13], resulting in the Klebanov-Witten superpotential,

$$W = Tr(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1).$$

Furthermore, it will turn out that the $SU(2) \times SU(2)$ symmetry of the resolved conifold will completely fix the superpotential terms involving the p and q fields, up to field redefinition.

Recall that the projective coordinates of the resolved conifold,

$$(x_1, x_2, y_1, y_2) \equiv (\lambda x_1, \lambda x_2, \lambda^{-1} y_1, \lambda^{-1} y_2),$$

transform with (x_1, x_2) in the doublet of $SU(2)_1$ and (y_1, y_2) in the doublet of $SU(2)_2$. In the above derivation of the Ext groups, it was clear that the n q and $(n-1)$ p fields live in various cohomology groups of the $\mathbb{P}^1$. These groups carry an induced action of the $SU(2)_1$ symmetry, which must be realized as a global symmetry group of the quiver quantum mechanics.

Therefore we conclude that q and p are in the $(\bar{n})$ and $(n-1)$ representation of the global $SU(2)_1$, under which A_1 and A_2 form a fundamental representation. There is a unique cubic superpotential that is invariant under this $SU(2)$, up to field definition, essentially because there is a single copy of the trivial representation in the tensor product $(\bar{n}) \otimes (n-1) \otimes (2)$. We first can construct from A and p a combination which is (n) under $SU(2)$. This is basically the same as constructing angular momentum states $|l+\frac{1}{2}, m+\frac{1}{2}\rangle$ from $|l, m\rangle$ and $|\frac{1}{2}, \pm\frac{1}{2}\rangle$, where $2l+1 = n-1$.

By using the following relation,

$$|l+\frac{1}{2}, m+\frac{1}{2}\rangle = \sqrt{\frac{l+m+1}{2l+1}}|l, m\rangle|\frac{1}{2}, \frac{1}{2}\rangle + \sqrt{\frac{l-m}{2l+1}}|l, m+1\rangle|\frac{1}{2}, \frac{-1}{2}\rangle, \tag{4.19}$$

we can write down explicitly the form of superpotential (ignoring the color trace and index structure and just focusing on the invariance of the $SU(2)$)

$$W \sim p_1 A_2 q_1 + (\sqrt{\frac{n-2}{n-1}} p_2 A_2 + \sqrt{\frac{1}{n-1}} p_1 A_1) q_2 + (\sqrt{\frac{n-3}{n-1}} p_3 A_2 + \sqrt{\frac{2}{n-1}} p_2 A_1) q_3 + \cdots \tag{4.20}$$

We should perform the following field redefinitions:

$$\tilde{q}_1 = q_1,\ \tilde{q}_2 = -\sqrt{\frac{1}{n-1}} q_2,\ \tilde{q}_3 = \sqrt{\frac{2}{(n-1)(n-2)}} q_3,\ \tilde{q}_4 = -\sqrt{\frac{3!}{(n-1)(n-2)(n-3)}} q_4,\ \cdots \tag{4.21}$$

The relations implied by superpotential become

$$A_1 \tilde{q}_2 = A_2 \tilde{q}_1,\ A_1 \tilde{q}_3 = A_2 \tilde{q}_2,\ \cdots, A_1 \tilde{q}_n = A_2 \tilde{q}_{n-1}. \tag{4.22}$$

Although these field redefinitions are not unitary and will spoil the D-terms, King's stability condition will not change under these redefinitions. The moduli space of solutions to the F-flatness and D-flatness conditions, modded out by the $U(N) \times U(M)$ gauge symmetry is equivalent to the $GL(N,\mathbb{C}) \times GL(M,\mathbb{C})$ quotient of the holomorphic F-term constraint, together with the King stability condition. Therefore, we can always bring the superpotential to the form we want, so that (4.3) holds.

4.5 Seiberg duality

Note that given one of the above quivers, all others can be constructed from it simply by repeated application of the rules developed by Berenstein and Douglas [16] for generalized Seiberg dualities. Suppose we begin with quiver Q_n. Then dualizing the $\mathcal{O}_C(-n-1)[1]$ node reverses the directions of all arrows through that node. In addition, between the other two nodes there will be $2n$ new mesonic fields $M_{ai} = A_a q_i$ for $a = 1, 2$, $i = 1, ..., n$.

The superpotential calculated above implies that there is a mass term $M_{ai} p_j$, where the indices are contracted as described above to be consistent with the $SU(2)$ flavor symmetry. This lifts all p_j and a corresponding $n-1$ of the mesons from the massless spectrum. Therefore, we obtain exactly the field content of the quiver Q_{n+1}! This result had to be true, given the previous calculation of the quiver directly from the new basis of objects in the derived category.

5 θ Stability and Cyclicity

The moduli space of supersymmetric Higgs branch vacua of the quiver quantum mechanics[6] describing the D6/D2/D0 bound states is given by the $U(N) \times U(M) \times U(1)$ Kähler quotient of the solution of the F-flatness conditions. The background values of the Kähler moduli are encoded in the values of the FI parameters in the Kähler quotient.

In general, it is a very difficult problem to determine the Euler character of the resulting moduli space, even by using the toric action to reduce to fixed points. We will find that for a particular choice of FI parameters the situation is dramatically simpler. This motivates

[6]Note that our quiver must be understood as a quantum mechanics, describing the BPS configurations of a point-like object in the $R^{3,1}$, rather than a $3+1$ field theory, as it would then be anomalous. This is obvious from the presence of a Calabi-Yau wrapping brane.

us to chooce a convenient basis of branes (that is, a particular mutation, Q_n, of the quiver) for which the FI parameters are of this simple type in a given chamber in the background Kähler moduli space.

It was shown by King [10] that is it possible to replace the D-term equation appearing in the Kähler quotient by a purely holomorphic algebraic condition, called θ-stability. Let $(\theta_v)_{v\in V}$ be the FI parameters, a set of real numbers assigned to the nodes of the quiver, such that $\theta(\mathbf{N}) = \sum N_v\theta_v = 0$ for a given dimension vector $\mathbf{N}$. Then a representation R is called θ-stable if for every *proper* subrepresentation $\tilde{R}$ with dimension vector $\tilde{\mathbf{N}}$, $\theta(\tilde{\mathbf{N}})$ is smaller than $\theta(\mathbf{N})$.

Consider the chamber in the space of FI parameters given by

$$\begin{aligned} &\theta_1 > 0,\ \theta_2 < 0,\ \theta_3 < 0, \quad \text{for } Q_n; \\ &\theta_1 > 0,\ \theta_2 < 0,\ \theta_3 < 0, \quad \text{for } \tilde{Q}_n. \end{aligned} \tag{5.1}$$

Our interest is in bound states with one unit of D6 charge, thus we have that $N_1 = 1$. Then King stability is equivalent to cyclicity, in the sense that the entire representation is generated by a vector in $\mathbb{C}$, the node associated to $\mathcal{O}_X[1]$. Firstly, any such representation is King stable for this choice of FI parameters, since any subrepresentation that includes this node must be the entire representation, and thus the proper subrepresentations all have $\theta(\tilde{\mathbf{N}}) = \tilde{N}_2\theta_2 + \tilde{N}_3\theta_3 < 0$.

Moreover, suppose that $\mathbf{R}$ is a King stable representation with $N_1 = 1$. Then consider the subrepresentation, $\tilde{\mathbf{R}}$, generated by the vector space $\mathbb{C}$ of the D6 node. If it is not all of $\mathbf{R}$, then $\tilde{N}_2 < N_2$ or $\tilde{N}_3 < N_3$, and one has that $\theta(\tilde{\mathbf{R}}) = \theta_1 + \tilde{N}_2\theta_2 + \tilde{N}_3\theta_3 > 0$, and the representation $\mathbf{R}$ must be unstable.

6 The big picture: connecting the dots

We would like to put together every piece of the story in this section. First of all, we observe that the D6/D2/D0 BPS partition function at a certain chamber, $\mathbb{C}_n$ in the Kähler moduli space is the same as the pyramid partition generating function for length n ERC, after the parameter identification (3.4),

$$Z_{D6/D2/D0}(u, v, C_n) = Z_{pyramid}(n; q_0, q_1). \tag{6.1}$$

By empirically checking the finite type pyramid partition, we also conjecture that the BPS states partition function is identical to the finite type pyramid partition generating function:

$$Z_{D6/D2/D0}(u, v, \tilde{C}_n) = Z_{finite}(n; q_0, q_1). \tag{6.2}$$

Physically, given a set of brane charges, we should be able to use the quiver theory to compute the Euler character of the moduli space. In order to do that, we need to know how to translate the data of chamber C_n or $\tilde{C}_n$ into the FI parameters of the corresponding quiver. This, in general, is a very difficult task.

In the conifold case, we are in luck because we have the answer from pyramid partition. We showed that pyramid partitions with length n ERC, as well as those of finite type, are torus fixed points in the moduli space of vacua of a certain quiver. Using this answer, we noticed that this quiver is Seiberg dual to the quiver with basis $\{\mathcal{O}_X[1], \mathcal{O}_C, \mathcal{O}_C(-1)[1]\}$. And in the Seiberg dual quiver, Q_n, we can determine the FI parameters to reproduce the cyclicity property. So we should keep in mind the following relation:

$$\begin{aligned} Z_{D6/D2/D0}(u, v, C_n) &= Z^{Q_0}_{quiver}(u, v, \theta^{Q_0}_i) \\ &= Z^{Q_n}_{quiver}(u, v, \theta^{Q_n}_1 > 0, \theta^{Q_n}_2 < 0, \theta^{Q_n}_3 < 0), \end{aligned} \tag{6.3}$$

$$\begin{aligned} Z_{D6/D2/D0}(u, v, \tilde{C}_n) &= Z^{Q_0}_{quiver}(u, v, \tilde{\theta}^{Q_0}_i) \\ &= Z^{\tilde{Q}_n}_{quiver}(u, v, \theta^{\tilde{Q}_n}_1 > 0, \theta^{\tilde{Q}_n}_2 < 0, \theta^{\tilde{Q}_n}_3 < 0). \end{aligned} \tag{6.4}$$

The quivers Q_n and $\tilde{Q}_n$ are Seiberg dual to the quiver Q_0, which has basis

$$\{\mathcal{O}_X[1],\ \mathcal{O}_C,\ \mathcal{O}_C(-1)[1]\}.$$

Presumably, we should be able to find the mapping:

$$C_n \leftrightarrow \theta^{Q_0}_i \leftrightarrow \theta^{Q_n}_i, \quad \tilde{C}_n \leftrightarrow \tilde{\theta}^{Q_0}_i \leftrightarrow \theta^{\tilde{Q}_n}_i. \tag{6.5}$$

For Q_n, the θ stability condition for $\{\theta^{Q_n}_1 > 0, \theta^{Q_n}_2 < 0, \theta^{Q_n}_3 < 0\}$ gives exactly the rules for constructing the pyramid partition in length n ERC. On the other hand, the θ

stability of the quiver $\tilde{Q}_n$ with $\{\theta_1^{\tilde{Q}_n} > 0, \theta_2^{\tilde{Q}_n} < 0, \theta_3^{\tilde{Q}_n} < 0\}$ gives the rules for constructing the finite type pyramid partition.

It is also possible to obtain the mapping (6.5) between the chambers in the space of Kähler moduli and the FI parameters before matching the answers. Consider the chamber $[\mathcal{W}_n^1 \mathcal{W}_{n+1}^1]$, which we checked corresponds to the simple choice of FI parameters for the quiver $\tilde{Q}_n$. This contains the locus $Im(z) = 0$, $-n-1 < Re(z) < -n$ along the boundary of the Kähler cone for $\pi/3 < \varphi < 2\pi/3$. The D2/D0 branes associated to the sheaves $\mathcal{O}_C(-n-1)[1]$ and $\mathcal{O}_C(-n-2)$ have charges $-\beta + ndV$ and $+\beta - (n+1)dV$.

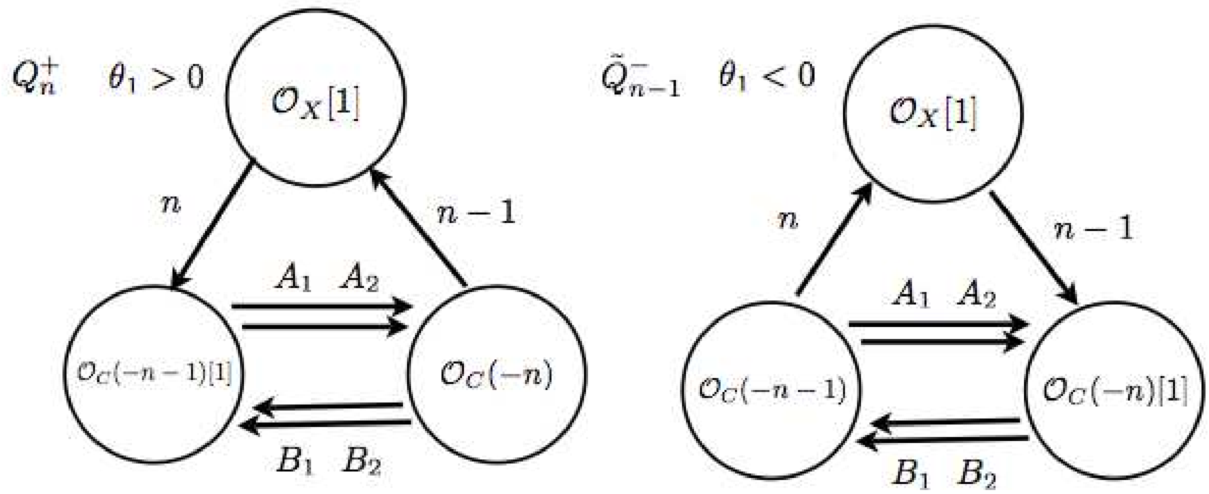

Figure 7: The quiver Q_n^+ and $\tilde{Q}_n^-$.

When their central charges $Z(-\beta + ndV; t) = -z - n$ and $Z(+\beta - (n+1)dV; t) = z + n + 1$ are aligned, then the bifundamental strings stretched between these two branes must become massless (at tree level) in the quiver quantum mechanics. Referring to the form of the bosonic potential, we see this occurs precisely when the FI parameters for those nodes are equal. Therefore we are in the chamber expected. This provides an *a priori* derivation of the partition function of D6/D2/D0 bound states in each chamber.

One last thing to notice is that we can flip the signs of the θs and the directions of the arrows of the quiver at the same time, without causing any change to the partition function of the quiver theory. The reason is that in this way we do not change the D-term conditions at all. Therefore, we have:

$$Z_{Q_n^+}(u, v) = Z_{\tilde{Q}_{n-1}^-}(u^{-1}, v), \quad Z_{Q_n^-}(u, v) = Z_{\tilde{Q}_{n-1}^+}(u^{-1}, v) \tag{6.6}$$

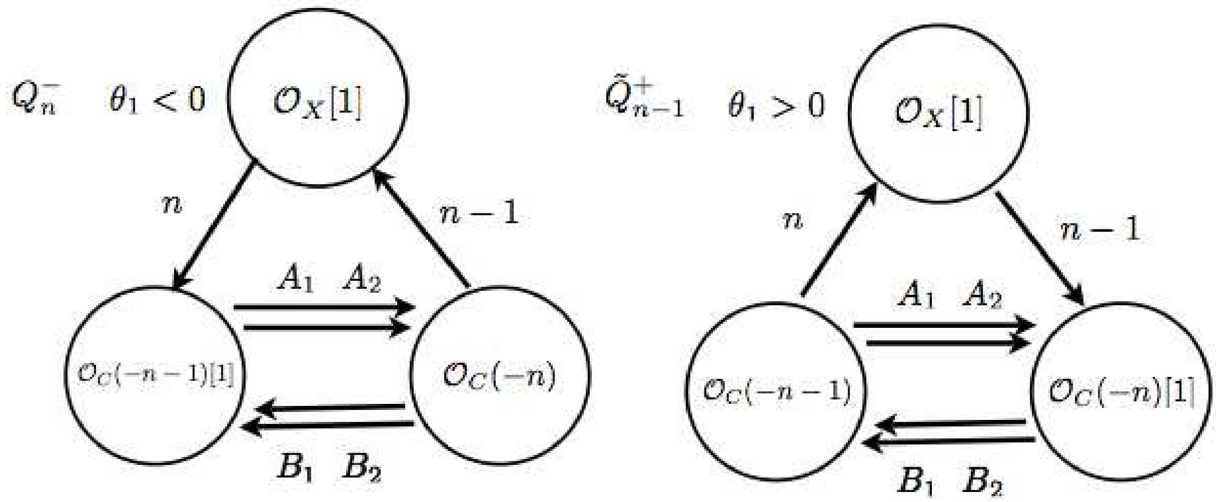

Figure 8: The quiver Q_n^- and $\tilde{Q}_n^+$.

where we simplify our notation by specifying the signs of θ_1 in the quiver by putting a superscript on the Q_n or $\tilde{Q}_n$.

7 Conclusion

In this paper we have studied the relation between the pyramid partition generating function and the D6/D2/D0 BPS state partition function on the resolved conifold. We found that the generating function of the pyramid partition with length n ERC is equivalent to D6/D2/D0 BPS state partition functions in certain chambers in the Kähler moduli space. More precisely, we have the following relation:

$$Z_{D6/D2/D0}(u, v, C_n) = Z_{pyramid}(n; q_0, q_1), \quad C_n = [\mathcal{W}_n^{-1}\mathcal{W}_{n-1}^{-1}], \tag{7.1}$$

$$Z_{D6/D2/D0}(u, v, \tilde{C}_n) = Z_{finite}(n; q_0, q_1), \quad \tilde{C}_n = [\mathcal{W}_{n-1}^1\mathcal{W}_n^1], \tag{7.2}$$

where the chambers $[\mathcal{W}_n^{-1}\mathcal{W}_{n-1}^{-1}]$ and $[\mathcal{W}_{n-1}^1\mathcal{W}_n^1]$ are defined in [7].

From the rules specifying pyramid partitions (of both infinite and finite type), we constructed the corresponding quivers, the θs parameters, and the superpotentials. We gave the underlying basis of sheaves and verified that they are Seiberg dual to the original D6/D2/D0 systems. The arrow structures of the quivers are also verified by computing Ext groups. The θ parameters in these particular basis are simple and the superpotentials

are quartic, so that the rules of pyramid partition emerge. We also noted that the cyclicity condition on quiver representations is the same as the King stability condition in the region of (5.1).

It would also be interesting to see if there is a similar story for the BPS partition function in the chambers other than $[\mathcal{W}_n^{-1}\mathcal{W}_{n-1}^{-1}]$ and $[\mathcal{W}_{n-1}^{1}\mathcal{W}_n^{1}]$. We also note that pyramid partitions with more colors have been developed in [17], and we suspect a similar story will emerge in the case of orbifold Donaldson-Thomas partition function.

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